Square-Primitive Gaps

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Introduction

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- Prime gaps are an intensive area of research, with breakthrough results as recently as last year. Unproven conjectures about the primes abound, such as the Twin Prime Conjecture.
- We will consider the sizes of gaps between successive terms in prime-like sequences.

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- The primes are an example of a primitive set.

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- $\Omega(n)$ is the number of prime factors of n counted with multiplicity.
- For example, $\Omega(2^5 3^3 5^1 7^2 11^1) = 5 + 3 + 1 + 2 + 1 = 12$.

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- $\Omega(mn) = \Omega(m) + \Omega(n)$
- Note that if $a|b$, $a \neq b$, then $b = ka$ and hence $\Omega(b) = \Omega(k) + \Omega(a) > \Omega(a)$.

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- $\Omega(mn) = \Omega(m) + \Omega(n)$
- Note that if $a|b$, $a \neq b$, then $b = ka$ and hence $\Omega(b) = \Omega(k) + \Omega(a) > \Omega(a)$.
- This shows that the set of positive integers n with $\Omega(n) = k$ for a given k is primitive. The case $k = 1$ corresponds to the primes.

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- An important example of a square-primitive set is the set of squarefree numbers.
- Note that all primitive sets are square-primitive, while not all square-primitive sets are primitive.

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- $$\{1, 2, 3, 5, 8, 13\}?$$
- $$\{3, 4, 10, 14, 17, 22\}?$$

A word about density

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- It can be shown that the density of the squarefree numbers is $\frac{6}{\pi^2}$.
- This contrasts with the primes, which have density 0. Furthermore, it can be shown that any primitive set with a well-defined density has density 0.

The Finite-Gap Problem

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Conjecture

There does not exist a square-primitive set S such that the gap sequence of S is bounded.

Squarefree Gaps

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Proof: Let $p_1, p_2, ..., p_m$ be the first m primes. By the Chinese Remainder Theorem, there exists a positive integer n such that $p_i^2 | n + i$ for $1 \leq i \leq m$. Hence, we can construct arbitrarily long intervals which contain no squarefree numbers, and we are done.

Main Theorem

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If S is square-primitive set, then the gap sequence of S contains infinitely many terms greater than 2.

Proof: Suppose that S is square-primitive and N is such that if $n > N$, either $n \in S$ or $n+1 \in S$. Note that if $n > N$, $n \in S$, and $n+1 \in S$, then

$$n(4n+3)^2 = 16n^3 + 24n^2 + 9n \notin S$$

and

$$(n+1)(4n+1)^2 = 16n^3 + 24n^2 + 9n + 1 \notin S$$

which is impossible. Hence, S contains either all even $n > N$ or all odd $n > N$. However, this forces S to contain either two distinct powers of 4 or two distinct powers of 9.

Directions Forward

- Use density arguments to bound size of bounded-gap square-primitive sets
- For example, a square-primitive set S with gaps of length at most 3 must contain one of $900n + 124$, $900n + 125$, $900n + 126$ hence must avoid one of $(900n + 124)/4 = 225n + 31$, $(900n + 125)/25 = 36n + 5$, $(900n + 126)/9 = 100n + 14$. We can use these types of facts to bound the density of S.

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- Probabilistic Method arguments to construct square-primitive sets with small gaps ($O(\log^2 n(\log\log n)^\epsilon)$ currently)
- Sieve Theory is often useful when dealing with these kinds of problems. For example, one can use Sieve Theory to prove the existence of primitive sets with gaps that are $O(n^\epsilon)$ for any given $\epsilon > 0$.

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